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A QUOTIENT OF $C(\omega^\omega)$ WHICH IS NOT ISOMORPHIC TO A SUBSPACE OF $C(\alpha)$, $\alpha < \omega_1$

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ABSTRACT

A quotient space of $C(\omega^\omega)$, the continuous functions on the ordinals not greater than ω^ω with the order topology, is constructed which is not isomorphic to a subspace of $C(\alpha)$, $\alpha < \omega_1$.

§0. Introduction

Johnson and Zippin [4] have shown that a quotient of c_0 is isomorphic to a subspace of c_0. (See also [1].) This and the fact that a quotient of $C[0,1]$ is isomorphic to a subspace of $C[0,1]$ gives rise to the question of whether or not this is a general property of separable $C(K)$ spaces, i.e., if X is a quotient of $C(K)$, for some compact metric space K is X isomorphic to a subspace of $C(K)$. A positive answer to this question would have shed additional light on the problem of classifying the complemented subspaces of $C[0,1]$ in view of the fact that each complemented subspace of $C[0,1]$ is a quotient of $C(\alpha)$, for some $\alpha < \omega_1$, or is isomorphic to $C[0,1]$, [2].

In §1 of this paper we show that there is a quotient X of $C_0(\omega^\omega)$ which is not isomorphic to a subspace of $C(\alpha)$, for any $\alpha < \omega_1$. In addition, X is not isomorphic to a complemented subspace of any $C(K)$ space and X^* is isometric to l_1.

We will use standard Banach space notation as may be found in [5]. If α is an ordinal, $C(\alpha)$ (resp. $C_0(\alpha)$) denotes the space of continuous functions on the ordinals not greater than α with the order topology (resp., and vanishing at α). We will denote the αth derived set of a topological space K by $K^{(\alpha)}$.

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Finally, we will without further comment consider elements of the spaces $C(\alpha)^*$, $\alpha < \omega_1$, as either functions defined on the ordinals or as measures.

§1. Our quotient space X of $C_0(\omega^\omega)$ will be a c_0 sum of quotients X_n of $C(\omega^{2n})$, $n = 1, 2, \cdots$. The following procedure is basic to the definition of X_n.

Let x be an element of the unit sphere of $l_1(\omega^n) = C(\omega^n)^*$, for some integer n, and suppose that

$$\operatorname{supp} x \subset [1, \omega^n]^{(m)} - [1, \omega^n]^{(m+1)}, \qquad m \geqq 2.$$

Let $\{N_i : i = 1, 2, \cdots\}$ be a partition of $\mathbf{N}$ into infinite sets, let $\{y_{2j+1} : j = 1, 2, \cdots\}$ be a norm dense set in the probability measures on $[1, \omega)$ such that the support of y_{2j+1} is finite for each j, and let $y_{2j} = e_j$, the jth vector of the usual unit vector basis of l_1. Define

$$x_{i(1)}(\alpha) = \begin{cases} x(\beta + \omega^m) y_{i(1)}(k) & \text{if } \alpha = \beta + \omega^{m-1}k, \text{ for some } \beta \in [1, \omega^n]^{(m)} \cup \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

$i(1) = 1, 2, \cdots$. Also for each $i(1)$, define elements $x_{i(1)i(2)}$, $i(2) = 1, 2, \cdots$ by

$$x_{i(1)i(2)}(\alpha) = \begin{cases} x_{i(1)}(\beta + \omega^{m-1}) & \text{if } \alpha = \beta + \omega^{(m-2)}k \text{ where } \beta \in [1, \omega^n]^{(m-1)} \text{ and } k \text{ is the } i(2)\text{th element of } N_{i(1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\|x_{i(1)}\|_1 = \|x_{i(1)i(2)}\|_1 = 1$, $w^* \lim_{i(2)\to\infty} x_{i(1)i(2)} = x_{i(1)}$, and if y is a w* cluster point of $\{x_{i(1)i(2)} : i(1), i(2) = 1, 2, 3, \cdots\}$ then

$$\begin{aligned} y &\in \overline{\text{co}}(\{x_{i(1)} : i(1) = 1, 2, \cdots\} \cup \{x\}) \\ &= \overline{\text{co}}(\{x_{i(1)} : i(1) = 2, 4, 6, \cdots\} \cup \{x\}). \end{aligned}$$

To define X_1, we take $m = n = 2$ and $x = \delta_{\omega^2}$. Then $x_{2l} = \delta_{\omega l}$, $l = 1, 2, \cdots$, each of the measures $x_{i(1)i(2)}$ is supported in $\{\omega k + l : l \in N_{i(1)}, k \in \mathbf{N}\}$ and

$$x_{i(1)i(2)}(\omega k + l) = \begin{cases} x_{i(1)}(\omega(k+1)) & \text{if } l \text{ is the } i(2)\text{th element of } N_{i(1)}, \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 0, 1, 2, \cdots.$$

Note that $\operatorname{supp} x_{i(1)i(2)} \cap \operatorname{supp} x_{i(1)'i(2)'} = \varnothing$ if $(i(1), i(2)) \neq (i(1)', i(2)')$. Let

$$Y_1 = \overline{\text{span}}\,(\{x_{i(1)i(2)} : i(1), i(2) = 1, 2, \cdots\} \cup \{x_{i(1)} : i(1) = 1, 2, \cdots\} \cup \{x\})$$

and X_1 be the predual of Y_1, i.e., if $(Y_1)_\perp = \{f \in C(\omega^2) : \langle y, f\rangle = 0 \text{ for all } y \in Y_1\}$, then $X_1 = C(\omega^2)/(Y_1)_\perp$. It is easy to see that Y_1 is w* closed and that a basis for Y_1, 1-equivalent the usual unit vector basis of l_1, is

$$\{x_{i(1)i(2)} : i(1), i(2) = 1, 2, \cdots\} \cup \{x_{2l} : l = 1, 2, \cdots\} \cup \{x\}.$$

To define X_2 we iterate this construction in the following way. Let $x = \delta_{\omega^4} \in l_1(\omega^4)$ and define the sequences $\{x_{i(1)} : i(1) = 1, 2, \cdots\}$ and $\{x_{i(1)i(2)} : i(1), i(2) = 1, 2, \cdots\}$ as above. Note that

$$\text{supp}\, x_{i(1)i(2)} \subset [1, \omega^4]^{(2)} - [1, \omega^4]^{(3)}$$

and we can use our procedure starting with $x_{i(1)i(2)}$ to define sequence $\{x_{i(1)i(2)i(3)} : i(3) = 1, 2, \cdots\}$ and $\{x_{i(1)i(2)i(3)i(4)} : i(3), i(4) = 1, 2, \cdots\}$. Precisely,

$$x_{i(1)i(2)i(3)}(\alpha) = \begin{cases} x_{i(1)i(2)}(\beta + \omega^2)y_{i(3)}(k) & \text{if } \alpha = \beta + \omega k \text{ for some} \\ & \quad \beta \in [1, \omega^4]^{(2)} \cup \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_{i(1)i(2)i(3)i(4)}(\alpha) = \begin{cases} x_{i(1)i(2)i(3)}(\beta + \omega) & \text{if } \alpha = \beta + k, \text{ some } \beta \in [1, \omega^4]^{(1)} \text{ and } k \\ & \quad \text{is the } i(4)\text{th element of } N_{i(3)}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_2 = \overline{\text{span}}\{x_t : t \text{ is a } n\text{-tuple of positive integers } 0 \leqq n \leqq 4\}$ and let $X_2 = C(\omega^4)/(Y_2)_\perp$. It is easy to see that Y_2 is w* closed and that

$$\{x_t : t \text{ is a } n\text{-tuple}, 0 \leqq n \leqq 4, \text{ and if } t \text{ is of odd length, the last entry is even}\}$$

is a basis for Y_2 equivalent to the usual unit vector basis of l_1.

Continuing in this way, we define for each n a w* closed subspace

$$Y_n = \overline{\text{span}}\{x_t : t \text{ is an } l\text{-tuple}, 0 \leqq l \leqq 2n\}$$

where $x = x_t, t$ is the 0-tuple, is $\delta_{\omega^{2n}}$. Clearly Y_n is isometric to l_1 and $X_n = C(\omega^{2n})/(Y_n)_\perp$ is its predual.

Obviously $X = (\Sigma X_n)_{c_0}$ is isometric to $(\Sigma C(\omega^{2n}))_{c_0}/(\Sigma(Y_n)_\perp)_{c_0}$, $(\Sigma C(\omega^{2n}))_{c_0}$ is isometric to $C_0(\omega^\omega)$, and X^* is isometric to $(\Sigma Y_n)_{l_1}$ which is isometric to l_1.

To show that X is not isomorphic to a subspace of $C(\alpha)$, for any $\alpha < \omega_1$, it is sufficient to show that for every $\delta > 0$ there is an integer n such that $B_{X_n^*}$ does not contain a countable set A homeomorphic to a closed subset of $[1, \alpha]$ which is δ

norming, i.e.,

$$\sup\{|\langle a, x\rangle| : a \in A\} \geqq \delta \|x\|.$$

This is equivalent to saying that $\delta^{-1}\overline{\text{co}} \pm A \not\supset B_{X_n^*} = B_{Y_n}$.

The main step in our argument is contained in the following lemma:

LEMMA. *Suppose A is a countable $\sigma(l_1(\omega), C(\omega))$ closed subset of $B_{l_1}(\omega)$. Then for every $\varepsilon > 0$ there is a probability measure x,* $\operatorname{supp} x \subset [1, w)$, *such that*

$$\|ax + by\| \geqq (|a| + |b|\|y\|)(1-\varepsilon) - |b|\varepsilon/4, \qquad \forall a, \quad b \in \mathbf{R}, \quad \forall y \in A.$$

Before proving the lemma we will use it to show that X is not isomorphic to a subspace of $C(\alpha)$, for any $\alpha < \omega_1$.

Fix n and define for each l-tuple t, $0 \leqq l \leqq 2n$, an operator $R_t : l_1(\omega^{2n}) \to l_1(\omega^{2n})$ by

$$R_t\mu = \mu_{|\cup\{(\alpha_r^-, \alpha_r] : r = 1, 2, \cdots, r_0\}},$$

where $\operatorname{supp} x_t = \{\alpha_r : r = 1, 2, \cdots, r_0\}$ and $\alpha_r^- = \operatorname{supp}\{\gamma : \gamma < \alpha_r$ and if $\alpha_r \in [1, \omega^{2n}]^{(m)}$, for some m, $1 \leqq m \leqq 2n$, $\gamma \in [1, \omega^{2n}]^{(m)} \cup \{0\}$. Note that if the length of t is even $R_t(Y_n) \subset Y_n$.

Also define, for each t, an operator T_t on $l_1(\omega^{2n})$ by

$$T_t\mu = \sum_{r=1}^{r_0} \mu(\alpha_r^-, \alpha_r]\delta_{\alpha_r}; \qquad T\mu = \mu[1, \omega^{2n}]\delta_{\omega^{2n}}.$$

It is easy to see that $T_t = T_tR_t$, $T_t(Y_n) \subset Y_n$, and if $t = (i(1), i(2), \cdots, i(l))$, $T_t x_{i(1), \cdots, i(j)} = x_t$, if $j \geqq l$. We leave it to the reader to check that both R_t and T_t are w* continuous.

Consider $U_1 = \sum_{l=1}^{\infty} T_{2l} + T[I - \sum_{l=1}^{\infty} R_{2l})$. (The infinite sums are convergent in the strong operator topology.) U_1 is w* continuous, $U_1(Y_n) \subset Y_n$, and if B is a subset of B_{Y_n} homeomorphic to $[1, \alpha]$, $U_1(B)$ is a countable w* closed subset of the measures supported on $\{\omega^{2n-1}l : l = 1, 2, \cdots\} \cup \{\omega^{2n}\}$. The space of measures on this set is clearly isometric to $l_1(\omega)$ and thus by the lemma there is an element y of norm one such that

$$\|ay + bx\| \geqq (1-\varepsilon)(|a| + |b|\|x\|) - |b|\varepsilon/4$$

for all $a, b \in \mathbf{R}$, and $x \in U_1(B)$ and such that $\operatorname{supp} y \subset \{\omega^{2n-1}l : l = 1, 2, \cdots\}$. Here ε is a fixed positive number.

Observe that

$$U_1(\{x_{i(1)} : i(1) = 1, 2, \cdots\}) = \{x_{i(1)} : 1(1) = 1, 2, \cdots\}$$

is norm dense in the probability measures supported on $\{\omega^{2^{n-1}}l : l = 1, 2, \cdots\}$. Thus we can assume that $y = x_{j(1)}$, for some integer $j(1)$. (Precisely speaking we should apply the lemma with some $\varepsilon' < \varepsilon$ and then choose $j(1)$ such that $\| x_{j(1)} - y \| < \varepsilon - \varepsilon'$.)

Next consider

$$U_2 = \sum_{l=1}^{\infty} T_{j(1),1,2l} + T_{j(1),1}\left(I - \sum_{l=1}^{\infty} R_{j(1),1,2l}\right).$$

Clearly U_2 is w* continuous and $U_2(Y_n) \subset Y_n$. As before $U_2(B)$ is a w* closed countable subset of a space isometric to $l_1(\omega)$ (namely, $\overline{\operatorname{span}}(\{x_{j(1),1,2l} : l = 1, 2, \cdots\} \cup \{x_{j(1),1}\})$) and thus by the lemma there is an element

$$y \in \overline{\operatorname{span}}\{x_{j(1),1,2l} : l = 1, 2, \cdots\}, \qquad \| y \| = 1,$$

such that

$$\| ay + bx \| \geqq (1 - \varepsilon)(|a| + |b| \| x \|) - |b| \varepsilon/4$$

for all $a, b \in \mathbf{R}$ and $x \in U_2(B)$. Since

$$U_2(\{x_{j(1),1,i(1)} : i(1) = 1, 2, \cdots\}) = \{x_{j(1),1,i(1)} : i(1) = 1, 2, \cdots\},$$

we can assume that $y = x_{j(1),1,j(3)}$, for some integer $j(3)$.

In this way we find indices $j(1), j(3), \cdots, j(2n-1)$ such that if

$$U_k = \sum_{l=1}^{\infty} T_{j(1),1,j(3),1,\cdots,j(2k-3),1,2l} + T_{j(1),1,j(3),1,\cdots,j(2k-3),1}\left(I - \sum_{l=1}^{\infty} R_{j(1),1,\cdots,j(2k-3),1,2l}\right),$$

then

$$\| ax_{j(1),1,j(3),1,\cdots,j(2k-1)} + bx \| \geqq (1 - \varepsilon)(|a| + |b| \| x \|) - |b| \varepsilon/4$$

for all $x \in U_k(B)$, $k = 1, 2, \cdots, n$.

Let

$$S_k = T_{j(1),1,\cdots,j(2k-1),1} + \sum_{l=1}^{k} U_l(I - R_{j(1),1,\cdots,j(2l-1),1}),$$

$k = 1, 2, \cdots, n$. We claim that if

$$x_{j(1),1,\cdots,j(2n-1),1} = \sum_{i=1}^{\infty} \lambda_i z_i, \qquad \{z_i : i \in \mathbf{N}\} \subset B,$$

then

$$\sum_{i=1}^{\infty} |\lambda_i| \| R_{j(1),1,\cdots,j(2r-3),1} S_k (I - R_{j(1),1,\cdots,j(2r-1),1}) z_i \| \geqq 2(1-\varepsilon) - \sum_{i=1}^{\infty} |\lambda_i| \varepsilon/4,$$

for $r = 1, 2, \cdots, k$ $(R = I)$.

We will establish this claim by induction on k. If $k = 1$, $S_1(\sum_{i=1}^{\infty} \lambda_i z_i) = x_{j(1),1}$. By our choice of $j(1)$

$$\| a x_{j(1)} + b U_1 z_i \| \geqq (1-\varepsilon)(|a| + |b| \| U_1 z_i \|) - |b| \varepsilon/4,$$

for all $a, b \in \mathbf{R}$, $i \in \mathbf{N}$. Let $T_{j(1),1} z_i = \gamma_i x_{j(1),1}$ and note that since

$$(S_1 z_i)_{|\operatorname{supp} x_{j(1),1}} = T_{j(1),1} z_i,$$

$\sum_{i=1}^{\infty} \lambda_i \gamma_i = 1$. We have that $U_1 x_{j(1),1} = x_{j(1)}$, so that letting $a = -\gamma_i$ and $b = 1$, we get that

$$\begin{aligned} \| U_1(I - R_{j(1),1}) z_i \| &= \| U_1 z_i - U_1 T_{j(1),1} z_i \| \\ &= \| U_1 z_i - \gamma_i U_1 x_{j(1),1} \| \\ &\geqq (1-\varepsilon)(|\gamma_i| + \| U_1 z_i \|) - \varepsilon/4. \end{aligned}$$

Summing over i, we have that

$$\begin{aligned} \sum_{i=1}^{\infty} |\lambda_i| \| S_1(I - R_{j(1),1}) z_i \| &= \sum_{i=1}^{\infty} |\lambda_i| \| U_1(I - R_{j(1),1}) z_i \| \\ &\geqq (1-\varepsilon)\left(\sum_{i=1}^{\infty} |\lambda_i| |\gamma_i| + \sum_{i=1}^{\infty} |\lambda_i| \| U_1 z_i \| \right) - \sum_{i=1}^{\infty} |\lambda_i| \varepsilon/4 \\ &\geqq 2(1-\varepsilon) - \sum_{i=1}^{\infty} |\lambda_i| \varepsilon/4, \end{aligned}$$

since $\sum_{i=1}^{\infty} \lambda_i \gamma_i = 1$ and $x_{j(1)} = U_1 x_{j(1),1} = \sum_{i=1}^{\infty} \lambda_i U_1 z_i$.

Now inductively assume we have proved our claim for $k-1$. Because

$$S_{k-1}(I - R_{j(1),1,\cdots,j(2k-3),1}) S_k + T_{j(1),1,\cdots,j(2k-3),1},$$

we have

$$\begin{aligned} &R_{j(1),1,\cdots,j(2r-3),1} S_{k-1}(I - R_{j(1),1,\cdots,j(2r-1),1}) \\ &\quad = R_{j(1),1,\cdots,j(2r-3),1} S_k (I - R_{j(1),1,\cdots,j(2r-1),1}), \end{aligned}$$

for $r = 1, 2, \cdots, k-1$ and thus we need only show that

$$\sum_{i=1}^{\infty}|\lambda_i|\,\|R_{j(1),1,\cdots,j(2k-3),1}S_k(I-R_{j(1),1,\cdots,j(2k-1),1})\|\geqq 2(1-\varepsilon)-\sum_{i=1}^{\infty}|\lambda_i|\,\varepsilon/4.$$

By the choice of $j(2k-1)$,

$$\|ax_{j(1),1,\cdots,j(2k-1)}+bU_kz_i\|\geqq(1-\varepsilon)(|a|+|b|\,\|U_kz_i\|)-|b|\,\varepsilon/4.$$

As before let

$$T_{j(1),1,\cdots,j(2k-1),1}z_i=(S_kz_i)_{|\operatorname{supp}x_{j(1),1,\cdots,j(2k-1),1}}$$

$$=\gamma_ix_{j(1),1,\cdots,j(2k-1),1}$$

and note that $\Sigma_{i=1}^{\infty}\lambda_i\gamma_i=1$.

$$U_kx_{j(1),\cdots,j(2k-1),1}=x_{j(1),1,\cdots,j(2k-1)}$$

and if we choose $a=-\gamma_i$ and $b=1$, we have that

$$\|U_k(I-R_{j(1),1,\cdots,j(2k-1),1})z_i\|=\|U_kz_i-\gamma_iU_kx_{j(1),\cdots,j(2k-1),1}\|$$

$$\geqq(1-\varepsilon)(|\gamma_i|+\|U_kz_i\|)-\varepsilon/4.$$

Summing over i we get

$$\sum_{i=1}^{\infty}|\lambda_i|\,\|U_k(I-R_{j(1),1,\cdots,j(2k-1),1})z_i\|$$

$$\geqq(1-\varepsilon)\left(\sum_{i=1}^{\infty}|\lambda_i|\,|\gamma_i|+\sum_{i=1}^{\infty}|\lambda_i|\,\|U_kz_i\|\right)-\sum_{i=1}^{\infty}|\lambda_i|\,\varepsilon/4$$

$$\geqq 2(1-\varepsilon)-\sum_{i=1}^{\infty}|\lambda_i|\,\varepsilon/4,$$

because $\Sigma_{i=1}^{\infty}\lambda_iU_kz_i=x_{j(1),1,\cdots,j(2k-1)}$. Observe that

$$R_{j(1),1,\cdots,j(2k-3),1}S_k(I-R_{j(1),1,\cdots,j(2k-1),1})=U_k(I-R_{j(1),1,\cdots,j(2k-1),1})$$

and thus the claim is proved.

Taking $k=n$ we have that

$$\sum_{i=1}^{\infty}|\lambda_i|\geqq\sum_{i=1}^{\infty}|\lambda_i|\,\|S_nz_i\|$$

$$\geqq\sum_{i=1}^{\infty}|\lambda_i|\left[\sum_{r=1}^{n}\|R_{j(1),1,\cdots,j(2r-3),1}S_n(I-R_{j(1),1,\cdots,j(2r-1),1})z_i\|\right.$$

$$\left.+\|R_{j(1),1,\cdots,j(2n-1),1}S_nz_i\|\right]$$

$$\geqq \sum_{r=1}^{n}\left[2(1-\varepsilon)-\sum_{i=1}^{\infty}|\lambda_i|\varepsilon/4\right]+1$$

$$=2n(1-\varepsilon)+1-\sum_{i=1}^{\infty}|\lambda_i|\varepsilon n/4.$$

Hence

$$\sum_{i=1}^{\infty}|\lambda_i| \geqq \frac{2n(1-\varepsilon)+1}{1+\varepsilon n/4}.$$

Because ε was arbitrary, it follows that the norming constant of X_n is at most $1/(2n+1)$. Thus $X=(\Sigma X_n)_{c_0}$ is not isomorphic to a subspace of $C(\alpha)$, for any $\alpha<\omega_1$.

Proof of the Lemma. By the Mazurkiewicz–Sierpinski theorem [6], A is homeomorphic to $[1,\omega^\gamma n]$, for some $\gamma<\omega_1$ and $n\in\mathbf{N}$. We will prove our result by inductionon γ and n. Thus we will assume as our inductive hypothesis that for all $\beta<\gamma$, $n\in\mathbf{N}$, and $\varepsilon>0$, if $A=\{x_\alpha:\alpha\leqq\omega^\beta n\}$, where $x_\alpha\to\alpha$ is a homeomorphism, and if $\{y_n:n\in\mathbf{N}\}$ is a sequence of unit vectors of finite support with $\operatorname{supp} y_n\cap\operatorname{supp} y_m=\emptyset$, for $n\neq m$, then there is an element $z=\sum_{n=1}^{\infty}a_ny_n$, $a_n\geqq 0$, $\sum_{n=1}^{\infty}a_n=1$, $a_n\neq 0$ for only finitely many n, such that for all a,b and $\alpha\leqq\omega^\beta n$,

$$\|az+bx_\alpha\|\geqq(1-\varepsilon)(|a|+|b|\|x_\alpha\|)-|b|\varepsilon/4.$$

First we will show that our inductive hypothesis actually will give us a sequence $\{z_i:i\in\mathbf{N}\}$ of disjointly supported convex combinations of $\{y_n:n\in\mathbf{N}\}$ such that for all $\{a_i:i\in\mathbf{N}\}\subset\mathbf{R}$, $b\in\mathbf{R}$ and $\alpha\leqq\omega^\beta n$,

$$\left\|\sum a_iz_i+bx_\alpha\right\|\geqq(1-\varepsilon)\left(\sum|a_i|+|b|\|x_\alpha\|\right)-|b|\varepsilon/4.$$

Let $\varepsilon_i\downarrow 0$, $\sum_{i=1}^{\infty}\varepsilon_i<\varepsilon/2$. By applying the inductive hypothesis for ε_i we can find a sequence $\{z_i:i\in\mathbf{N}\}$ such that $\operatorname{supp} z_i$ is finite, $\operatorname{supp} z_i\cap\operatorname{supp} z_j=\emptyset$, $i\neq j$, and for all $a,b\in\mathbf{R}$ and $\alpha\leqq\omega^\beta n$,

$$\|az_i+bx_\alpha\|\geqq(1-\varepsilon_i)(|a|+|b|\|x_\alpha\|)-|b|\varepsilon_i/4.$$

It follows that

$$\begin{aligned}\|a_iz_i+bx_{\alpha|\operatorname{supp} z_i}\|&\geqq(1-\varepsilon_i)(|a_i|+|b|\|x_\alpha\|)-|b|\varepsilon_i/4-|b|\|x_{\alpha|\operatorname{supp} z_i{}^C}\|\\&=(1-\varepsilon_i)(|a_i|+|b|\|x_{\alpha|\operatorname{supp} z_i}\|)\\&\quad-|b|\varepsilon_i/4-\varepsilon_i|b|\|x_{\alpha|(\operatorname{supp} z_i)^C}\|.\end{aligned}$$

Summing over i we get that

$$\left\| \sum_{i=1}^{\infty} a_i z_i + b x_\alpha \right\| \geqq \sum_{i=1}^{\infty} (1-\varepsilon_i)(|a_i| + |b| \| x_{\alpha|\mathrm{supp}\, z_i} \|)$$

$$+ |b| \| x_{\alpha|(\cup\{\mathrm{supp}\, z_i : i \in \mathbf{N}\})^c} \| - |b| \sum_{i=1}^{\infty} \varepsilon_i \| x_\alpha \| - |b| \sum_{i=1}^{\infty} \varepsilon_i / 4$$

$$\geqq (1-\varepsilon)\left(\sum_{i=1}^{\infty} |a_i| + |b| \| x_\alpha \| \right) - |b| \| x_\alpha \| \varepsilon/2 - |b| \varepsilon/8$$

$$\geqq (1-\varepsilon)\left(\sum_{i=1}^{\infty} |a_i| + |b| \| x_\alpha \| \right) - |b| \varepsilon/4.$$

The inductive hypothesis is obvious for $\gamma = 0$. Suppose $A = \{x_\alpha : \alpha \leqq \omega^\gamma\}$ for some $\gamma \geqq 1$. Choose n_0 such that $\| x_{\omega^\gamma [n_0, \omega)} \| < \varepsilon^3/128$ and assume (without loss of generality) that $\mathrm{supp}\, y_l \subset [n_0, \omega)$ for all l. Let

$$A_1 = \{\alpha : \| x_{\alpha|\mathrm{supp}\, y_1} \| \geqq \varepsilon^3/64\}.$$

A_1 is w* closed and does not contain ω^γ. Hence by the inductive hypothesis there are elements $\{z_{1i} : i \in \mathbf{N}\}$ which are finite convex combinations of $\{y_n : n > 1\}$, disjointly supported, and for all $\alpha \in A_1$,

$$\left\| \sum_{i=1}^{\infty} a_i z_{1i} + b x_\alpha \right\| \geqq (1 - \varepsilon^3/64)\left(\sum_{i=1}^{\infty} |a_i| + |b| \| x_\alpha \| \right) - |b| \varepsilon^3/512.$$

Let $A_2 = \{\alpha : \| x_{\alpha|\mathrm{supp}\, z_{11}} \| \geqq \varepsilon^3/64\}$ and as above let $\{z_{2i} : i \in \mathbf{N}\}$ be a sequence of finite convex combinations of $\{z_{1i} : i > 1\}$ which are disjointly supported and satisfy

$$\left\| \sum_{i=1}^{\infty} a_i z_{2i} + b x_\alpha \right\| \geqq (1 - \varepsilon^3/64)\left(\sum_{i=1}^{\infty} |a_i| + |b| \| x_\alpha \| \right) - |b| \varepsilon^3/512$$

for all $\alpha \in A_2$.

By repeating this argument $l = 4/\varepsilon - 1$ times (we may assume $4/\varepsilon \in \mathbf{N}$), we get elements $\{z_{ji} : i \in \mathbf{N},\ j = 0, 1, 2, \cdots, l\}$ (let $y_i = z_{0i}$, $i = 1, 2, \cdots$) such that

(a) $\{z_{ji} : i \in \mathbf{N}\}$ is a sequence of finite convex combinations of $\{z_{j-1,i} : i > 1\}$,

(b) $\mathrm{supp}\, z_{ji} \cap \mathrm{supp}\, z_{jk} = \varnothing$, $i \neq k$,

(c) $\mathrm{supp}\, z_{ji} \cap \mathrm{supp}\, z_{j-1,1} = \varnothing$, $\forall i \in \mathbf{N}$, $j = 1, 2, \cdots, l$,

(d) if $\alpha \in A_j = \{\alpha : \| x_{\alpha|\mathrm{supp}\, z_{j-1,1}} \| \geqq \varepsilon^3/64\}$,

$$\left\| \sum_{i=1}^{\infty} a_i z_{ji} + b x_\alpha \right\| \geqq (1 - \varepsilon^3/64)\left(\sum_{i=1}^{\infty} |a_i| + |b| \| x_\alpha \| \right) - |b| \varepsilon^3/512.$$

Let $z = \frac{1}{4}\varepsilon \sum_{j=0}^{l} z_{j1}$ and fix $\alpha \leqq \omega^{\gamma}$:

$$\|az + bx_\alpha\| = \left\| \frac{a\varepsilon}{4} \sum_{j=0}^{l} z_{j1} + bx_\alpha \right\|$$

$$= \left\| \frac{a\varepsilon}{4} \sum_{j=0}^{j_0-2} z_{j1} + bx_{\alpha \mid \cup\{\operatorname{supp} z_{j1} : j \leqq j_0 - 2\}} \right\|$$

$$+ \left\| \frac{a\varepsilon}{4} z_{j_0-1,1} + bx_{\alpha \mid \operatorname{supp} z_{j_0-1,1}} \right\|$$

$$+ \left\| \frac{a\varepsilon}{4} \sum_{j=j_0}^{l} z_{j1} + bx_{\alpha \mid (\cup\{\operatorname{supp} z_{j1} : j < j_0\})^c} \right\|,$$

where j_0 is the first integer j for which $\alpha \in A_j$. For $j \leqq j_0 - 2$, $\|x_{\alpha \mid \operatorname{supp} z_{j1}}\| < \varepsilon^3/64$, so that the first term is larger than

$$(j_0 - 1)\frac{|a|\varepsilon}{4} - (j_0 - 1)\frac{|b|\varepsilon^3}{64}.$$

For $j \geqq j_0$, each z_{j1} is a convex combination of $\{z_{j_0,i} : i > 1\}$, and thus by (d), the last term is larger than

$$(1 - \varepsilon^3/64)\left(\frac{|a|\varepsilon}{4}(l - j_0 + 1) + |b|\,\|x_{\alpha \mid (\cup\{\operatorname{supp} z_{j1} : j < j_0\})^c}\|\right)$$

$$- \frac{|b|\varepsilon^3}{64}\|x_{\alpha \mid \cup\{\operatorname{supp} z_{j1} : j < j_0\}}\| - \frac{|b|\varepsilon^3}{512}.$$

Finally,

$$\left\| \frac{a\varepsilon}{4} z_{j_0-1,1} + bx_{\alpha \mid \operatorname{supp} z_{j_0-1,1}} \right\| \geqq -\frac{|a|\varepsilon}{4} + |b|\left\| x_{\alpha \mid \operatorname{supp} z_{j_0-1,1}} \right\|.$$

Hence

$$\|az + bx_\alpha\| \geqq \frac{|a|\varepsilon}{4}(j_0 - 1) - \frac{|b|\varepsilon^3}{64}(j_0 - 1) - \frac{|a|\varepsilon}{4} + |b|\,\|x_{\alpha \mid \operatorname{supp} z_{j_0-1,1}}\|$$

$$+ (1 - \varepsilon^3/64)\left(\frac{|a|\varepsilon}{4}(l - j_0 + 1) + |b|\,\|x_{\alpha \mid (\cup\{\operatorname{supp} z_{j1} : j < j_0\})^c}\|\right)$$

$$- \frac{|b|\varepsilon^3}{64}\|x_{\alpha \mid \cup\{\operatorname{supp} z_{j1} : j < j_0\}}\| - \frac{|b|\varepsilon^3}{512}$$

$$\geqq \frac{|a|\varepsilon}{4}(l - 1)(1 - \varepsilon^3/64) + |b|(1 - \varepsilon^3/64)\|x_{\alpha \mid (\cup\{\operatorname{supp} z_{j1} : j < j_0 - 1\})^c}\|$$

$$-\frac{|b|\varepsilon^3}{64}(j_0-1)-\frac{|b|\varepsilon^3}{64}\|x_\alpha\|-\frac{|b|\varepsilon^3}{512}$$

$$\geqq |a|(1-\varepsilon/2)(1-\varepsilon^3/64)+|b|(1-\varepsilon^3/64)\|x_\alpha\|$$

$$-|b|(1-\varepsilon^3/64)\|x_{\alpha|\cup\{\operatorname{supp} z_{j1}:j<j_0-1\}}\|-\frac{|b|\varepsilon^3}{64}(j_0+1)$$

$$\geqq (1-\varepsilon)|a|+(1-\varepsilon)|b|\|x_\alpha\|-\frac{|b|\varepsilon^3}{64}(j_0-1)-\frac{|b|\varepsilon^3}{64}(j_0+1)$$

$$\geqq (1-\varepsilon)|a|+(1-\varepsilon)|b|\|x_\alpha\|-|b|\varepsilon^2/8$$

$$\geqq (1-\varepsilon)(|a|+|b|\|x_\alpha\|)-|b|\varepsilon/4,$$

establishing the lemma for γ. Clearly a similar argument will yield the case $n>1$.

Finally, we wish to show that X is not isomorphic to a complemented subspace of any $C(K)$ space. We will accomplish this by showing that the l_1-predual Z constructed (beginning with the one dimensional space) by Benyamini and Lindenstrauss [3] is isomorphic to a complemented subspace of X. Since Z is not complemented in a $C(K)$ space, the result will follow.

Suppose that in our sequence $\{y_{2j-1}: j=1,2,\cdots\}$ we have $y_1=\frac{1}{2}\delta_1+\frac{1}{2}\delta_2$, $y_3=\frac{2}{3}\delta_1+\frac{1}{3}\delta_2$, and $y_5=\frac{1}{3}\delta_1+\frac{2}{3}\delta_2$. (These vectors may not have been in our original sequence, but we can assume they are without affecting the preceding argument.) In Y_n consider the subspace

$Z_n=\overline{\operatorname{span}}\{x_t: t$ is an l-tuple, $0<l<2n$, with l odd, all of whose odd entries except the last are 1, 3, or 5, and the last is either 2 or 4$\}$.

(Recall that $y_2=\delta_1$, $y_4=\delta_2$.) It is easy to see that Z_n is w* closed and isometric to l_1. (The defining set given above is a basis.) Also we leave it to the reader to check (by induction) that Z_n is w* isometric to the dual of the nth space of Benyamini and Lindenstrauss.

It remains to show that Z_n is w* complemented in Y_n with the norm of the projection independent of n. To do this we employ a selection theorem for L_1-preduals (theorem II.4.17 of [5]). Define $\psi: B_{Y_n}\to 2^{B_{Z_n}}$ by

$$\psi(y)=y_{|\operatorname{supp} Z_n}+\|y_{|(\operatorname{supp} Z_n)^c}\|B_{Z_n}$$

where $\operatorname{supp} Z_n=\bigcup\{\operatorname{supp} z: z\in Z_n\}$. It is easy to see that ψ is lower semicontinuous (in the w* topology), symmetric, and convex. Therefore, there is a convex, symmetric, w* continuous selection $P: B_{Y_n}\to B_{Z_n}$ such that $P(y)\in\psi(y)$

for all $y \in B_{Y_n}$. Clearly P extends to a norm one w* continuous linear operator from Y_n to Z_n which is a projection onto Z_n.

REMARK. It is possible but messy to explicitly write down a projection from Y_n to Z_n. For example for $n = 2$, define

$$P_2 = (T_1 + T_4)\left(I - \sum_{k=1}^{2} \sum_{l=1}^{3} \sum_{i(2)=1}^{\infty} R_{(2l-1),i(2),2k}\right) + \sum_{k=1}^{2} \sum_{l=1}^{3} \sum_{i(2)=1}^{\infty} T_{(2l-1),i(2),2k}.$$

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